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Probabilistic Analysis of Composite Structure Ultimate Strength

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A practical system reliability-based technique is proposed for probabilistic estimation of the ultimate strength of composite structures. Important component failure sequences to system failure are identified using a fast branch and bound search procedure. Ply-level failure modes such as matrix cracking, fiber failure, and delamination are considered, and system degradation due to progressive failure and load shedding is modeled. The system failure probability is computed using the dominant failure sequences. The correlation between failure sequences is exploited to achieve computational efficiency. The variation of system failure probability with accumulating damage and load shedding is investigated to define the critical damage accumulation. The proposed methodology is applied to the analysis of a composite wing structure.

I. Introduction

COMPOSITE materials are being widely used in modern structures, such as aircraft and space vehicles, due to high performance and light weight. Considerable research on the design and failure analysis of composite structures is being conducted. The results of experimental and analytical studies show that composite materials have large statistical variations in their mechanical properties. Therefore, probabilistic analysis has to play an important role in structural assessment. This paper presents a finite-element-based procedure for the estimation of system reliability of composite structures and applies it to a composite aircraft wing analysis. The proposed method assesses the structural strength by combining progressive failure analysis and system reliability techniques.

Composite laminate failure may be considered in two major stages, first ply failure (FPF) and last ply failure (LPF). FPF usually corresponds to the commencement of matrix cracking failure; structural ultimate failure or LPF consists of a series of the ply-level component failures, such as matrix cracking, delamination, and fiber breakage, from the first one to last one. Methods such as the first-order reliability method (FORM) or the second-order reliability method (SORM) may be combined with finite element analysis to compute the component failure probability. Several earlier studies, for example, Refs. 1 and 2, have computed the FPF probability using FORM. However, because there is significant residual capacity after the FPF, this paper pursues the system-level reliability computation technique to assess the ultimate strength.

In this paper, a composite structure is treated as a system, and ply-level failure modes, namely, matrix cracking, fiber breakage, and delamination, are treated as components. The system failure is caused by a series of ply-level component failures. Generally, there exists a large number of failure sequences consisting of these components. Of these, a few dominant sequences may be identified, and the system failure probability may be approximated by the probability of union of the dominant sequences.

The branch and bound method may be employed to search for the significant system failure sequences. Earlier attempts[3] at using the system reliability technique have used the ply dropoff technique, where the entire ply strength is disregarded after a component failure such as matrix cracking. This paper considers the residual strength of the ply after matrix cracking. When each component failure occurs, the structural stiffness is modified to account for this damage, and the damaged structure is reanalyzed. This proceeds until system failure occurs. Based on the identified significant failure sequences, the system failure probability is determined by means of bounding techniques.

Such a strategy has been extensively applied to frame and truss structures, but becomes computationally time consuming for continuum structures such as an aircraft wing with numerous finite elements. However, in composite structures, multiple failure sequences tend to be highly correlated. Even within a single sequence, high correlations exist among several component failures.[4,5] This paper proposes a fast branch and bound technique that includes the correlation information between component failures and failure sequences to enumerate rapidly the significant failure sequences in composite structures.

Thus, this paper develops a practical method for the ultimate strength failure probability estimation of composite laminated structures. Several strategies are proposed for improving the efficiency of such computation. The correlation between component failures is exploited to develop a fast branch and bound search procedure. Ply-level failure modes such as matrix cracking, fiber failure, and delamination are considered, and system degradation due to progressive failure and load shedding is modeled. The system failure probability is computed using the dominant failure sequences to achieve computational efficiency. The correlation between failure sequences is used to develop a weakest link approximation. The variation of system failure probability with accumulating damage and load shedding is investigated to define the critical damage accumulation. The proposed method is applied to the ultimate strength failure probability analysis of a composite aicraft wing.

II. Probabilistic Progressive Failure Analysis

Three principal ideas form the basis of progressive failure and load-shedding analysis of composite structures:

1) When a ply sustains failure, its stiffness in corresponding directions is reduced. The damaged ply continues to contribute to the overall stiffness of the laminate unless it has completely delaminated from the rest of the structure. Thus, the stiffness matrix is modified corresponding to the type of ply-level failure, and the structure is reanalyzed for increasing loading until the next failure. The process is repeated until total failure of the entire laminate.

2) Under the influence of external loads, composite laminates suffer permanent loss of integrity or damage due to the formation of microcracks. The stress analysis of a composite laminate with thousands of small cracks is intractable. Alternatively, a procedure that takes the effect of microcracks into account in an average sense is desirable. The damaged material with many microcracks may be replaced with an equivalent material of degraded properties.

3) Macro-level strength failure criteria, such as maximum stress/strain and those of Tsai and Wu,[6] Hoffman,[7] Lee,[8] Tan,[9] Reddy and Reddy,[10] etc., are widely used in progressive failure analysis. In this paper, Lee's[8] simple criteria are used to derive the

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component-level performance functions for the sake of illustration, as follows.

For fiber breakage:

$$g_f = 1 - \sigma_1/X_T, \qquad \sigma_1 > 0$$

$$g_f = 1 + \sigma_1/X_C, \qquad \sigma_1 < 0 \tag{1}$$

For matrix cracking:

$$g_m = 1 - \left(\sigma_2^2/Y_T Y_C + \sigma_{12}^2/S^2\right) \tag{2}$$

For delamination:

$$g_d = 1 - (1/S_z)^2\left(\sigma_{23}^2 + \sigma_{31}^2\right) \tag{3}$$

In the preceding equations, X_T and X_C are the tensile and compressive strengths in the fiber direction, Y_T and Y_C are the tensile and compressive strengths in the direction transverse to fibers, and S and S_z are the shear strengths in x–y plane and in z direction, respectively. In Eqs. (1–3), $g < 0$ indicates the failure of the component, $g > 0$ indicates the survival of the component, and $g = 0$ is referred to as the limit state.

When fiber failure occurs, the ply stiffness terms contributed by the fiber (related to material direction 1) are modified to account for fiber breakage. That is, the moduli E_{11}, G_{12}, and G_{31} are reduced to zero. When matrix cracking occurs, the ply stiffness terms related to material direction 2 are modified to reflect matrix cracking. That is, the moduli E_{22}, G_{12}, and G_{23} are reduced to zero. When delamination failure occurs, the ply stiffness terms related to material direction 3, that is, the moduli G_{23} and G_{31}, are reduced to zero. In this paper, ANSYS is used for finite element analysis. The values of the relevant moduli are changed in the ANSYS datafile to model the corresponding failure during progressive failure analysis.

The proposed method for probabilistic progressive failure analysis uses the first-order shear deformation theory for laminated orthotropic plates. That is, normals to the centerplane are assumed to remain straight after deformation, but not necessarily normal to the centerplane. The ply stresses along the material axes can be obtained using finite element analysis. Then, they are substituted into a strength failure criterion for the reliability analysis of ply-level component failure modes.

The proposed analysis proceeds in the following manner. The basic random variables are determined first, which usually include material properties, strength properties, loads, ply orientations, and ply thicknesses. These are input to the structural stress analysis, for example, a finite element code. The computed stress results are inserted into component performance functions to estimate the failure probabilities of components. The component with higher failure probability is assumed to fail before the one with lower failure probability. Component failure is modeled by changing the corresponding terms in the stiffness matrix before reanalysis in the next iteration. The damaged structure is reanalyzed, and more failures are considered progressively, until the entire system collapses. Thus, a failure sequence is enumerated. Because numerous failure sequences are possible, the branch and bound method may be used to enumerate the significant failure sequences, as follows.

The branch and bound method has two operations. In the branching operation, starting from the intact structure, failure is imposed at the most likely location as indicated by the reliability analysis of all of the components. The structure is reanalyzed with the imposed failure, and the next failure is imposed at the location with the highest path probability. This process is repeated until a complete failure sequence is obtained. The bounding operation discards insignificant failure sequences, that is, sequences with path probability lower than a specified cutoff value γP_f, where γ is chosen by the analyst based on the required accuracy and P_f is the system failure probability. Because P_f is unknown, it is replaced by the maximum path probability P_f^* among the failure sequences already identified. The higher the value of γ, the smaller is the number of significant sequences enumerated. After this enumeration, the probability of system failure is computed as the probability of union of the significant sequences, as described in the next section.

III. Failure Probability Computation

FORM/SORM may be used to compute the component failure probability.[11] The probabilistically significant failure sequences are identified using the branch and bound method, described in Sec. IV. System failure probability may be estimated through the union of the significant failure sequences.[12]

In the reliability evaluation, the stress resultants $\{N\}$, the stress couples $\{M\}$, and the strength parameters X_T, X_C, Y_T, Y_C, and S are considered as the basic random variables. These basic random variables are expressed as a vector $\boldsymbol{X} = \{X_1, X_2, \ldots, X_m\}^T$. In FORM, the component reliability index β corresponding to a failure mode is computed as $\beta = (\boldsymbol{y}^{*T}\boldsymbol{y}^*)^{1/2}$, where $\boldsymbol{y}^*$ is the point of minimum distance from the origin to the limit state $g = 0$ in the $\boldsymbol{Y}$ space, where $\boldsymbol{Y}$ is the vector of equivalent uncorrelated standard normal variables and y^* is also referred to as the most probable point (MPP). The original random variables $\boldsymbol{X}$ may have any continuous distribution, and some of them may have statistical correlation. Several methods are available to transform the variables from $\boldsymbol{X}$ space to $\boldsymbol{Y}$ space.[11] In FORM, the first-order estimate of the failure probability is computed as $P(g \le 0) = \Phi(-\beta)$, where Φ is the cumulative distribution function of a standard normal variable. In SORM, several methods are available for the second-order estimate of the failure probability, making use of the curvature of the limit state at the MPP.

Once a significant failure sequence is identified by the branch and bound method and the corresponding component failure probabilities are computed, the next task is to compute the probability of failure of the overall sequence. If a failure sequence contains m individual failure components, the failure probability P_f^k of the kth failure sequence can be computed as the probability of intersection of its basic failure components, that is,

$$P_f^k = P\left(\cap_{j=1}^m E_j^k\right) \tag{4}$$

where E_j^k is the jth basic failure event in the kth failure sequence under the condition that the first $(j-1)$ basic failure events have occurred.

The computation of Eq. (4) is complicated when the number of components is large. Approximate bounding techniques are available for this purpose. The second-order upper bound suggested by Murotsu[13] is an option for the estimation of the joint failure probability. It may be expressed as

$$P\left(\cap_{j=1}^m E_j\right) \le \min_{i \ne j}[P(E_i \cap E_j)] \tag{5}$$

Several other alternative second-order upper bounds are available in the literature.[14,15] Also, higher-order bounds can be computed using the multinormal integral method developed by Gollwitzer and Rackwitz.[16] Monte Carlo sampling may also be used to compute the joint failure probability. For the case of two linear component limit states, the two-event joint probability in Eq. (5) may be computed as

$$P(E_i \cap E_j) = \Phi(-\beta_i, -\beta_j, \rho_{ij}) \tag{6}$$

where Φ is the normal distribution function, β_i and β_j are the reliability indices, and ρ_{ij} is the correlation coefficient computed as

$$\rho_{ij} = \sum_{r=1}^{n} \alpha_{ir}\alpha_{jr} \tag{7}$$

where α_{ir} and α_{jr} are the components of the unit gradient vectors of the limit states i and j.

If the component limit states are nonlinear, the two limit state functions are linearized with respect to their intersection, which is found from the constrained minimization problem:

$$\text{minimize} \quad \sqrt{\boldsymbol{Y}^T\boldsymbol{Y}}$$

$$\text{such that} \quad G_1(\boldsymbol{Y}) = 0, \qquad G_2(\boldsymbol{Y}) = 0 \tag{8}$$

Various optimization schemes can be used to solve this problem; the optimization feature available in the ANSYS finite element software (used for structural analysis) is used in this paper to perform the preceding optimization. Next, the correlation coefficient ρ can be obtained by the product of the unit vectors using Eq. (7). Then, Eq. (6) can be used to evaluate the joint failure probability.

IV. Proposed Method

The method outlined in Secs. II and III is applicable in general to the probabilistic ultimate strength analysis of any structure. In the past, it has been applied to frame and truss structures. However, this procedure is computationally time consuming for large structures with many components. In the case of an aircraft wing where the structural analysis is carried out through a finite element code with numerous finite elements, such a method is quite tedious. However, in the case of composite structures, the brittleness of the failures and the correlation between different failures and sequences can be used advantageously to introduce several techniques and approximations that make the method efficient and practical. These are investigated in this section.

A. Fast Branch and Bound Method

When there are a large number of components in a structural system, which is the case for composite structures, the basic branch and bound method described in Sec. II becomes time consuming and tedious in searching for important failure sequences. In the original method, only one component failure is imposed at each damaged stage so that a large number of steps is required to complete a failure sequence. This makes the basic branch and bound method difficult to apply in practice.

The following strategy may be used to speed up the enumeration procedure. The different component limit states share many common random variables related to loading and material properties. Therefore, there is correlation among the component limit state functions in any structure. This implies that, if a component fails, then other components that are highly correlated with this component may also fail subsequently with high probability. Therefore, in the failure sequence enumeration, several component failures (instead of only one failure) may be imposed together as a group at each damage stage.

For a group of strongly correlated components, the component having the highest failure probability is selected to be representative of the group at the current damage state. The failure probability for the group of components is approximated by the failure probability of the representative component. All of the components in the group are removed at the same time, and the structure is reanalyzed with the remaining components. The other steps in the proposed method are the same as that in the branch and bound method. This concept of grouping can drastically reduce the number of damage states and, hence, the number of structural reanalysis. Therefore, this method may be referred to as a fast branch and bound method. The savings in computational effort by using this strategy were studied by Xiao and Mahadevan[5] for framed structures. It was found that for even a small problem with 18 possible component failures, the fast branch and bound method requires only 0.6% of the computational time taken by the original branch and bound method. The savings in computational time grows with the size of the problem. The procedure is described as follows.

At any damage state, assume that $k-1$ components have failed and been removed and that $n-(k-1)$ components are still intact. The component k needs to be explored. A conditional probability criterion is checked for the other $n-k$ components as

$$P(E_i/E_k) \geq \lambda_0, \qquad i = k+1, \ldots, n \tag{9}$$

where λ_0 is a given cutoff value; E_k is the event of failure of the component k, E_i is the event of failure in the ith component among the remaining $n-k$ components, and $P(E_i/E_k)$ is the probability of event E_i given that the event E_k has occurred.

If Eq. (9) is satisfied for some components, then the components are selected into a group that fails together at the next stage. The failure probability of this group is approximately equal to the largest value among the component failure probabilities in the group.

To reduce the computational time in evaluating the conditional probability in Eq. (9), an alternative formula was proposed by Xiao and Mahadevan[5] as

$$P(E_i) \geq \lambda_0 P(E_k), \qquad i = k+1, \ldots, n \tag{10}$$

where n is the number of limit states considered and $P(E_i)$ and $P(E_k)$ are the probabilities of the individual events E_i and E_k, computed using FORM.[11] When the value of ρ_{ik}, the correlation coefficient between the ith and kth limit states, is unity (perfect correlation), Eq. (9) becomes Eq. (10). For cases where ρ_{ik} is less than 1, Eq. (10) selects a larger failure domain than Eq. (9) because $P(E_i)$ is always greater than $P(E_i \cap E_k)$. In other words, the limit states that satisfy Eq. (9) are a fraction of those that satisfy Eq. (10). The filtering through Eq. (10) helps to avoid the estimation of the joint probability integral for many unnecessary combinations of E_i and E_k.

The higher the value of λ in Eq. (9), the lower is the number of component failures for grouping in the fast branch and bound method. Thus, for $\lambda = 1$, the fast branch and bound method will become the original branch and bound method. Conversely, a lower value of λ will add more component failures for grouping. Some numerical investigation may be necessary for each structure to determine the optimum value of λ. A value of $\lambda = 0.5$ may be used as an initial guess.

B. Deterministic Initial Screening

Any component failure could be the starting point of a failure sequence. However, in the branch and bound enumeration, only component failures with high probability of occurrence are generally the starting points of the dominant failure sequences. The computation of component failure probabilities is described later. A performance function g is defined for each component failure in terms of all of the relevant random variables, such that $g < 0$ represents failure, $g > 0$ represents survival, and $g = 0$ is referred to as the limit state. The computation of the probability $P(g < 0)$ corresponding to each component failure involves several iterations of structural analysis to find the minimum distance point, as described subsequently. For large structures with numerous components, this first step in failure sequence enumeration is quite time consuming. Therefore, an efficient idea is to observe that the components with higher failure probabilities are in general likely to have the g values closer to zero, when the structure is analyzed at the mean values of the random variables. Therefore, the starting points of the dominant failure sequences may be selected using deterministic structural analysis at the mean values and by choosing only those component failures with g values below a cutoff value. Note that this strategy is simply to select the starting points of the sequences, not for the final probability computation. Therefore, it will provide significant reduction in computational effort, with minimal impact on the accuracy of the probability result.

Note that the probability-based criterion for all component failures is much more rigorous and accounts for the variation in random variable sensitivities. That is, some component failures that may appear insignificant with a deterministic criterion may become significant with a probabilistic criterion. Therefore, the deterministic screening should be done carefully, so that probabilistically significant events do not get discarded. In this paper, this screening is done as follows. The limit state value, calculated at the mean values of random variables, is used for selecting the 20 most likely component failures at each stage. (In the numerical examples considered in this paper, the components below the top 20 have very low failure probabilities.) Then, the 20 component failures are subjected to probabilistic analysis, and only the component failure with the highest probability among these 20 is used to start the fast branch and bound enumeration of the most probable failure sequence. Thus, the failure sequence enumeration is in fact probabilistic, except for the initial screening to reduce the computational effort.

C. Weakest Link Model

In the case of composite laminates, numerical studies[4] have shown that the various significant failure sequences are very similar in the list of failed components, which implies that there are strong correlations among these failure sequences. For example, consider a laminate as shown in Fig. 1, under plane stress loading. The significant failure sequences are enumerated and shown in Fig. 2. The numbers in the parentheses in Fig. 2 refer to the probabilities of the failure events. It is seen that the sequences are very similar, and the system failure probability may be approximated using the most probable sequence. Thus, it becomes reasonable to assume the weakest link model as valid for laminated composites. In that case, only one significant failure sequence might provide an adequate estimate of system failure probability. Therefore, instead of enumerating many failure sequences, it may be adequate to stop after the first significant failure sequence is identified. This provides tremendous savings in the computational effort of structural reanalysis corresponding to numerous steps of progressive failure in multiple sequences.

D. Critical Component Failure

Strictly speaking, a structural system failure is defined to be the collapse of the entire structure. Previous investigation[4] has shown that a failure sequence from initial failure to system final failure involves a large number of ply-level components. Thus, the search for even a single complete sequence is quite time consuming. Therefore, an approximate procedure is proposed here to avoid the tedious and expensive computation.

It is assumed that there exists a critical component for the structure. This critical component should have a failure probability much lower than the other components that fail after the critical component. The components with higher failure probabilities will easily fail once the critical one fails. In other words, the critical component failure is not far from the entire system failure. As a result, the structural system failure can be approximately defined to be the failure of the critical component.

The assumption of a critical failure to approximate system failure is quite reasonable for composite structures. The overall probability of the sequence is computed as the probability of intersection of the component events in that sequence. The probability of intersection of several events is dominated by the low probability events,[12] which are usually present before the first critical failure. After this critical failure, the probabilities of subsequent failures are much higher, thus not having a significant contribution to the probability computation. As an example, consider a plate structure made of composite laminates, as shown in Fig. 3. The probabilities of component failures along the failure sequence are shown in Fig. 4. It is seen that after the first fiber failure (which has the lowest probability), the

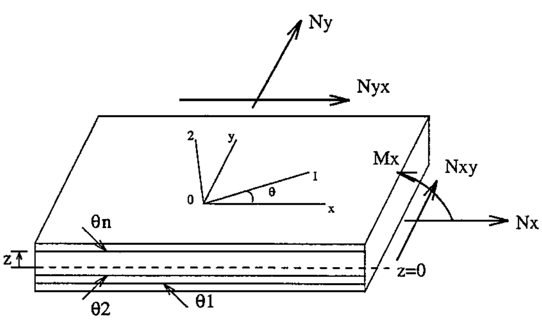

Fig. 1 Composite laminate.

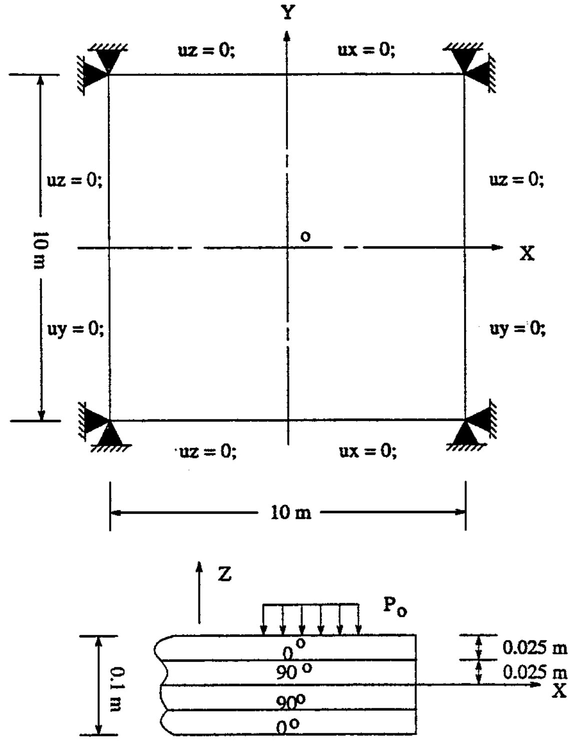

Fig. 3 Composite plate with transverse loading.

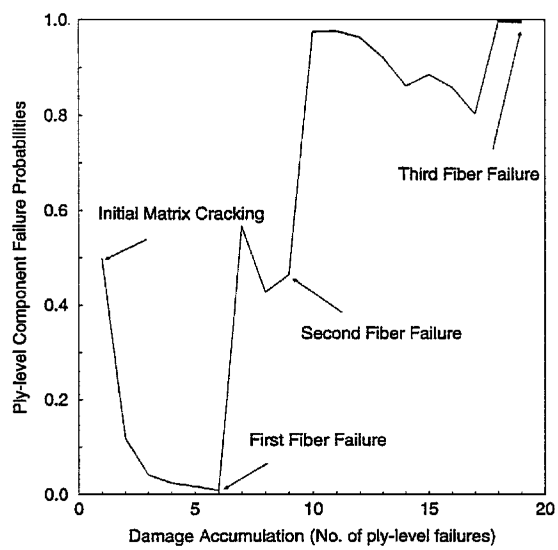

Fig. 4 Ply-level failure probabilities vs progressive damage.

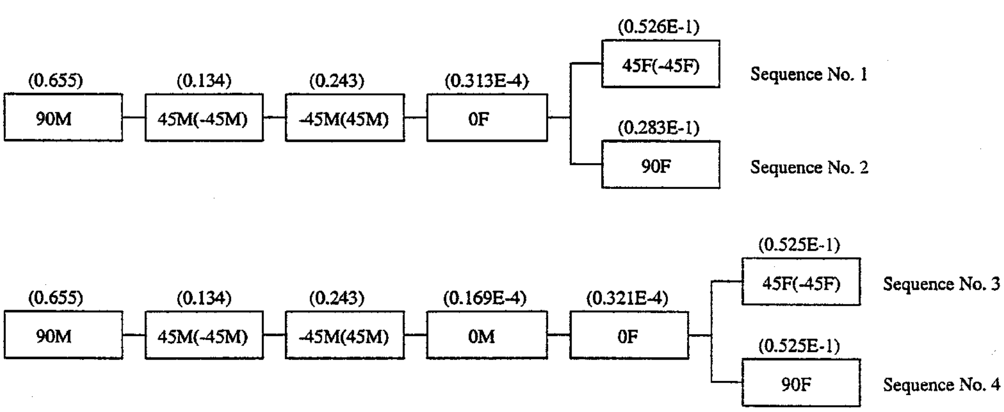

Fig. 2 Significant failure sequences for the laminate.

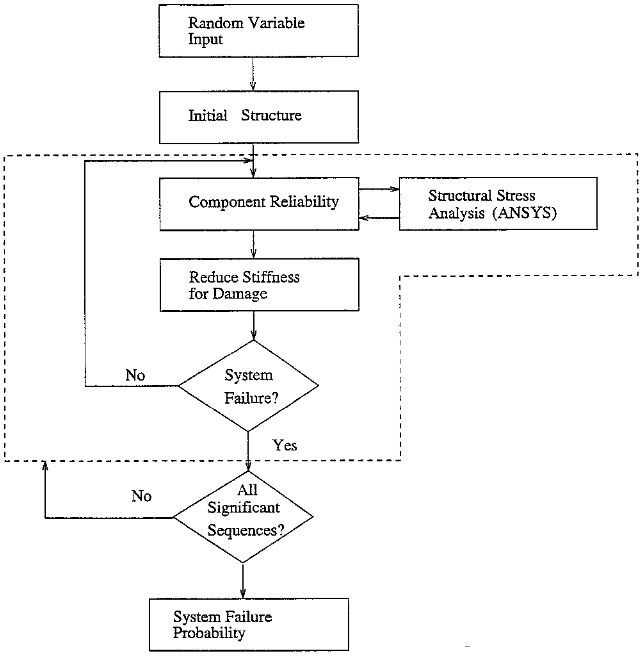

Fig. 5 Flowchart of the proposed method.

probabilities of the subsequent events are significantly increased. The overall probability of the sequence is dominated by the low probability events up to the first fiber failure and is closely approximated by the probability of intersection of the first fiber failure and a few matrix cracking failures immediately preceding it.[17] Therefore, considering probability computation, it appears adequate to terminate the exploration of the failure sequence at the first critical failure.

With the preceding assumption, a large amount of computation after the failure of the critical component can be avoided, and the prediction of ultimate failure probability will have adequate accuracy. The grouping concept and the weakest link model of the preceding two subsections provide additional computational efficiency. Thus, the proposed method may be summarized as follows:

1) Start from the intact structure. Calculate the values of performance functions (g functions) for all of the components and sort the values in descending order. Select the first several components with small g values to be explored (20 in this paper), using a suitable cutoff value for g.

2) Estimate the component failure probability of the selected components and arrange them in the descending order of probability values. Select the component with the highest failure probability and other components with high correlation to this first component.

3) Simulate the damage in the selected components by modifying the corresponding structural stiffness terms, and reanalyze the damaged structure to compute the failure probabilities of the remaining components.

4) Repeat steps 2 and 3 until the critical component fails. A flowchart of the proposed method is shown in Fig. 5.

V. Composite Aircraft Wing Analysis

An aircraft composite wing composed of skin and stringer components and consisting of center, leading edge, and trailing edge is shown in Fig. 6. The stringer is constructed in both longitudinal and transverse directions. Figure 6 describes the structural geometry and loading of a composite wing, and the data are taken from Ref. 1, where a probabilistic analysis of composite aircraft wing has been reported. The first ply failure criteria were assumed in that study. In this paper, the objective is to estimate the ultimate strength failure probability.

In Fig. 6, the left end (section A–A) is fixed and the right end (section B–B) is free. The air pressure is assumed to be triangularly distributed along transverse direction of the wing and linearly distributed along the longitudinal direction. The ply-level stiffness properties, material strengths, ply thicknesses and orientations, and pressure loads are all assumed to be random variables. Their mean values are indicated in Table 1. A laminate configuration of [0/−45/90/45/0 deg] is used for the skin and (0 deg)$_5$ for the stringer. The statistical distribution of a variable is assumed to be lognormal for the strength parameters and normal for the rest, with a coefficient of variation of 0.10. A standard deviation of 2 deg is used for the ply angles. The structure is modeled in ANSYS with 171 elements, 120 for skin and 51 for stringer, as shown in Fig. 7.

Table 1 Structural variables

Variable	Symbol	Mean value
Elastic modulus	E_{11}	12.80 msi (88252.93 MPa)
Elastic modulus	E_{22}	0.87 msi (5998.44 MPa)
Elastic modulus	E_{33}	0.87 msi (5998.44 MPa)
Shear modulus	G_{12}	0.55 msi (3792.12 MPa)
Shear modulus	G_{23}	0.46 msi (3171.59 MPa)
Shear modulus	G_{31}	0.46 msi (3171.59 MPa)
Poisson ratio	μ_{12}	0.26
Tensile strength; Eq. (1)	X_t	400.00 ksi (2757.90 MPa)
Compressive strength; Eq. (1)	X_c	400.00 ksi (2757.90) MPa
Tensile strength; Eq. (2)	Y_t	15.00 ksi (103.42 MPa)
Tensile strength; Eq. (2)	Y_c	35.00 ksi (241.32 MPa)
Shear strength; Eq. (2)	S_{12}	13.00 ksi (89.63 MPa)
Shear strength; Eq. (3)	S_z	13.00 ksi (89.63 MPa)
Ply thickness of skin	h_{sk}	0.02 in. (0.51 mm)
Ply thickness of stringer	h_{st}	0.08 in. (2.03 mm)
Peak pressure load (section A–A)	q_A	8.00 psi (55.16 kPa)
Peak pressure load (section B–B)	q_B	7.20 psi (49.64 kPa)

Table 2 Progressive damage of the composite aircraft wing

Damage state	Failed components	Representative failure probability
First	$85_{2M}, 85_{4M}, 86_{4M}, 86_{2M}$	0.2820
Second	$37_{2M}, 37_{4M}, 87_{4M}, 38_{2M}, 73_{2M}, 97_{4M}$	0.0903
Third	$87_{2M}, 49_{4M}, 97_{2M}, 25_{4M}, 98_{2M}$	0.0451
Fourth	$98_{4M}, 25_{2M}, 50_{4M}, 49_{2M}, 26_{4M}$ $73_{4M}, 26_{2M}, 38_{4M}$	0.0338
Fifth	$74_{2M}, 85_{3M}, 39_{2M}, 74_{4M}, 88_{4M}$ $89_{2M}, 88_{2M}, 86_{3M}$	0.0156
Sixth	$99_{4M}, 39_{4M}, 50_{2M}$	0.0036
Seventh	$51_{4M}, 13_{2M}, 13_{4M}$	0.0029
Eighth	$37_{3M}, 14_{4M}, 75_{2M}, 14_{2M}, 27_{4M}$ $85_{1F}, 89_{4M}, 85_{5F}$	0.0010

The methods of Sec. IV are used for efficient system reliability analysis. For the fast branch and bound method, $\lambda = 0.4$ is used in this example, only for the sake of demonstration. The choice of λ has to be based on a tradeoff between accuracy and efficiency. A larger value of λ reduces the number of failures in the grouping operation, and, therefore, makes the computation more expensive, but more accurate. For specific applications, the variation of failure probability estimate and computational effort with λ may be investigated, and an optimum value may be chosen.

The first significant failure sequence identified using the fast branch and bound method is used for ultimate strength failure probability estimation, using the weakest link model. The following notation is used to identify the ply-level failures: for example, 85_{2M} stands for finite element number 85, ply number 2, and matrix cracking failure mode; 37_{1F} stands for finite element number 37, ply number 1, and fiber breakage failure mode. The computed results are summarized in Table 2.

At each stage of damage, a few highly correlated components are chosen to fail together. Structural failure proceeds through progressive damage accumulation as shown in Table 1. Damage accumulates in the zones near the fixed end and at the center of the skin. The structure experiences eight stages of damage before experiencing the first fiber failure. In the damage process, the failure probability of each damage stage decreases with damage accumulation up to first fiber breakage occurrence. Probabilistic analysis is

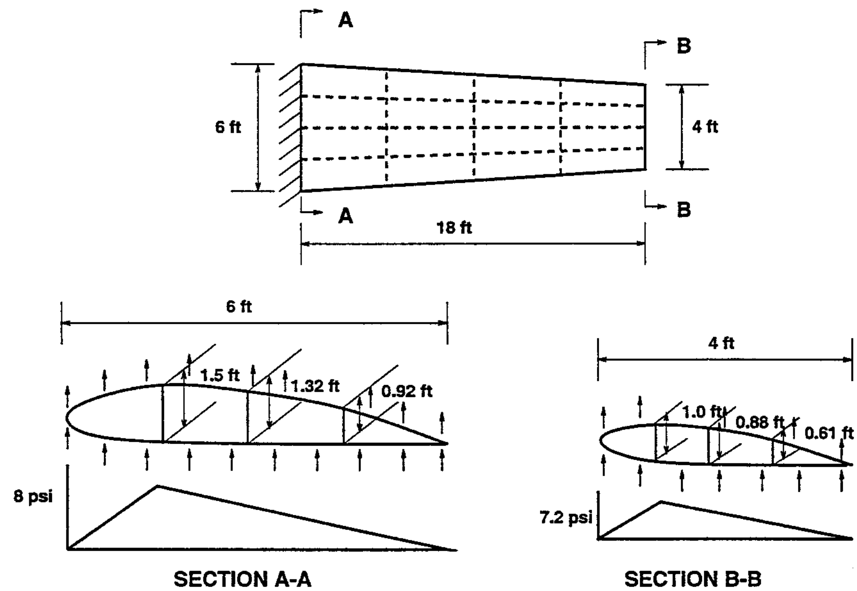

Fig. 6 Composite aircraft wing: geometry and loading.

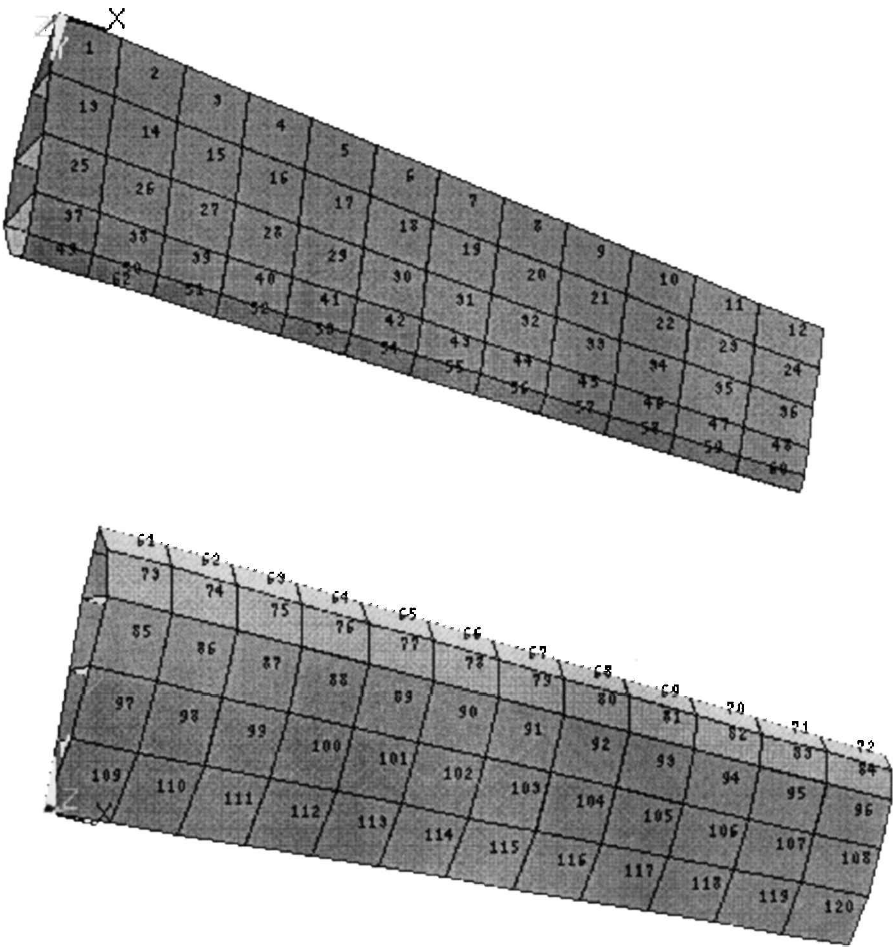

Fig. 7 Composite aircraft wing: finite element model.

also carried out for the ninth damage stage. Its representative failure probability increases to 0.028, which is much higher than the one before it. The variation of failure probability with the number of component failures in shown in Fig. 8. It is seen that the failure probability for the eighth damage stages is lowest, and at this stage the first fiber breakage occurs. After this event, the failure probabilities of the remaining components increase to significantly higher values. This indicates that the first fiber failure is likely to be a severe or critical failure for the entire structure. After this critical event, the structure will rapidly proceed to final failure. This implies that the exploration of the failure sequence may be terminated after this first severe failure.

The approximation of system failure with first fiber failure may appear too conservative from a deterministic point of view. However, there is a very good reason to terminate the failure sequence at this stage, due to the computation of intersection probability, as

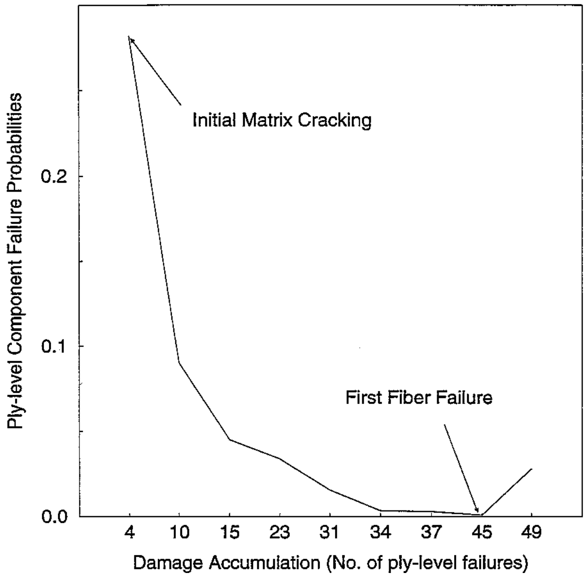

Fig. 8 Ply level failure probabilities with progressive damage.

mentioned in Sec. IV.D. In the present problem, the events corresponding to the sixth, seventh, and eighth damage stages are the most dominant. It is seen that failures after the first fiber failure have much higher probability, and, therefore, will not affect the intersection probability computation. Therefore, there is no need to explore the sequence any further, for the example considered here. For other structures, if the first fiber failure assumption is considered too conservative, the failure sequence may be explored further to see if there are other failure events with lower probabilities.

Three damage levels with the lowest failure probabilities, namely, the sixth, seventh, and eighth damage levels, dominate the computation of the system failure probability. The computed correlation coefficient matrix between the limit states representing these three events is

$$[\rho_{ij}] = \begin{bmatrix} 1 & \ldots & \text{sym} \\ 0.99 & 1 & \ldots \\ 0.71 & 0.70 & 1 \end{bmatrix} \tag{11}$$

The corresponding individual event failure probabilities and the two-event joint failure probabilities are obtained as

$$[P_{ij}] = \begin{bmatrix} 0.0036 & \ldots & \text{sym} \\ 0.0026 & 0.0029 & \ldots \\ 0.0003 & 0.0003 & 0.001 \end{bmatrix} \tag{12}$$

where P_{ij} refers to the joint probability of the ith and jth failures (the off-diagonal terms in the matrix). The diagonal terms in the matrix, $i = j$, refer to the individual failure probabilities.

Using only the two-event joint probabilities, the second-order upper bound formula suggested by Murotsu[13] is used to estimate the system failure probability as

$$P_f = 2.977 \times 10^{-4} \tag{13}$$

Using the three-event joint probability, the third-order estimate for the system failure probability is obtained as

$$P_f = 2.957 \times 10^{-4} \tag{14}$$

These two results are quite close to each other, as expected. For this reason, most system reliability studies report only second-order estimates.

VI. Conclusions

In this paper, a system reliability analysis procedure is developed for the probabilistic ultimate strength estimation of composite structures and is applied to the analysis of a composite aircraft wing. The fast branch and bound method, using the grouping concept, is used to speed up the search for the significant failure sequences. The failure sequences show high correlation, making it feasible to use the first significant sequence for a good approximation to the system failure probability. The system failure probability is approximated through a second-order bound of the probability of this significant sequence. Because the intersection probability of the sequence is dominated by a few low-probability events, a critical component failure, such as fiber failure, may be identified for a composite structure and system failure definition could be approximated as the occurrence of this event. These practical approximations may provide a good balance between accuracy, efficiency, and conservatism.

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